

Cross Power Density Spectrum

Young W Lim

November 8, 2019

Copyright (c) 2018 Young W. Lim. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

This work is licensed under a Creative Commons "Attribution-NonCommercial-ShareAlike 3.0 Unported" license.



Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Sum of two random processes

N Gaussian random variables

Definition

$$W(t) = X(t) + Y(t)$$

$$R_{WW}(t, t + \tau) = E[W(t)W(t + \tau)]$$

$$= E[\{X(t) + Y(t)\}\{X(t + \tau) + Y(t + \tau)\}]$$

$$= R_{XX}(t, t + \tau) + R_{YY}(t, t + \tau) + R_{XY}(t, t + \tau) + R_{YX}(t, t + \tau)$$

Sum of two random processes

N Gaussian random variables

Definition

$$W(t) = X(t) + Y(t)$$

$$R_{WW}(t, t + \tau) = E[W(t)W(t + \tau)]$$

$$= R_{XX}(t, t + \tau) + R_{YY}(t, t + \tau) + R_{XY}(t, t + \tau) + R_{YX}(t, t + \tau)$$

$$S_{WW}(\omega) = S_{XX}(\omega) + S_{YY}(\omega)$$

$$+ \mathcal{F} \{A[R_{XY}(t, t + \tau)]\} + \mathcal{F} \{A[R_{YX}(t, t + \tau)]\}$$

Sum of two random processes

N Gaussian random variables

Definition

$$W(t) = X(t) + Y(t)$$

$$R_{WW}(t, t + \tau) = E[W(t)W(t + \tau)]$$

$$= E[\{X(t) + Y(t)\}\{X(t + \tau) + Y(t + \tau)\}]$$

$$= R_{XX}(t, t + \tau) + R_{YY}(t, t + \tau) + R_{XY}(t, t + \tau) + R_{YX}(t, t + \tau)$$

Truncated Ensemble Members

N Gaussian random variables

Definition

$$x_T(t) = \begin{cases} x(t) & -T < t < +T \\ 0 & \textit{elsewhere} \end{cases}$$

$$y_T(t) = \begin{cases} y(t) & -T < t < +T \\ 0 & \textit{elsewhere} \end{cases}$$

$$x_T(t) \iff X_T(\omega)$$

$$y_T(t) \iff Y_T(\omega)$$

Cross Power

N Gaussian random variables

Definition

$$\begin{aligned}P_{XY}(T) &= \frac{1}{2T} \int_{-T}^{+T} x_T(t)y_T(t)dt \\ &= \frac{1}{2T} \int_{-T}^{+T} x(t)y(t)dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{2T} X_T^*(\omega)Y_T(\omega)d\omega\end{aligned}$$

Average and Total Average Cross Power

N Gaussian random variables

Definition

$$\begin{aligned}\bar{P}_{XY}(T) &= \frac{1}{2T} \int_{-T}^{+T} R_{XX}(t, t) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{2T} E[X_T^*(\omega) Y_T(\omega)] d\omega \\ P_{XY} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} R_{XX}(t, t) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} E[X_T^*(\omega) Y_T(\omega)] d\omega\end{aligned}$$

Cross Power Density Spectrum

N Gaussian random variables

Definition

$$S_{XY}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E[X_T^*(\omega) Y_T(\omega)]$$

$$P_{XY} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XY}(\omega) d\omega$$

$$S_{YX}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E[Y_T^*(\omega) X_T(\omega)]$$

$$P_{YX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{YX}(\omega) d\omega$$

Cross Power Density Spectrum

N Gaussian random variables

- 1 $S_{XY}(\omega) = S_{YX}(-\omega) = S_{YX}^*(\omega)$
- 2 $\Re[S_{XY}(\omega)]$ and $\Re[S_{YX}(\omega)]$ are even function of ω
- 3 $\Im[S_{XY}(\omega)]$ and $\Im[S_{YX}(\omega)]$ are odd function of ω
- 4 $S_{XY}(\omega) = 0$ and $S_{YX}(\omega) = 0$ if $X(t)$ and $Y(t)$ are orthogonal
- 5 if $X(t)$ and $Y(t)$ are uncorrelated and have constant mean \bar{X} and \bar{Y} , then $S_{XY}(\omega) = S_{YX}(\omega) = 2\pi\bar{X}\bar{Y}\delta(\omega)$
- 6 $A[R_{XY}(t, t + \tau)] \iff S_{XY}(\omega)$
- 7 $A[R_{YX}(t, t + \tau)] \iff S_{YX}(\omega)$

for jointly wide-sense stationary processes

N Gaussian random variables

Definition

$$S_{XY}(\omega) = \int_{-\infty}^{+\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$$

$$S_{YX}(\omega) = \int_{-\infty}^{+\infty} R_{YX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XY}(\omega) e^{+j\omega\tau} d\omega$$

$$R_{YX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{YX}(\omega) e^{+j\omega\tau} d\omega$$

for jointly wide-sense stationary processes -DT

N Gaussian random variables

Definition

$$S_{XY}(e^{j\Omega}) = \sum_{k=-\infty}^{+\infty} R_{XY}[k]e^{-j\Omega k}$$

$$R_{XY}[k] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} S_{XY}(e^{j\Omega})e^{+j\Omega k} d\Omega$$

